"Generalized" Harmonic Coordinates Using Abigel

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Harmonic Coordinates

- Harmonic coordinates, x^μ satisfy the condition

$$\Gamma^\lambda \equiv g^{\mu\nu}\Gamma^\lambda_{\mu\nu} = 0 \qquad \longmapsto \qquad \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\kappa}\left(\sqrt{-g}g^{\lambda\kappa}\right) = 0$$

such that each x^μ satisfies the wave equation $\Box x^\mu = 0$
With the d'Alembertian, $\Box\phi \equiv \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\lambda}\left(\sqrt{-g}g^{\lambda\kappa}\frac{\partial\phi}{\partial x^\kappa}\right)$

- One of the advantages of harmonic coordinates is that it gives a simplified form of the covariant wave equation in a curved spaces time that contains only second derivatives:

$$\Box^2\phi \equiv g^{\lambda\kappa}\frac{\partial^2\phi}{\partial x^\lambda \partial x^\kappa} - \Gamma^\lambda\frac{\partial\phi}{\partial x^\lambda} = 0 \qquad \left(because \qquad \Gamma^\lambda \equiv 0\right)$$

"Generalized" Harmonic Coordinates

An extension of harmonic coordinates, referred to as 'generalized' harmonic slicing is defined by $\Box x^b = F^b$. With F^b as a source function chosen to provide more flexibility and possibly avoid problems encountered with $F^b = 0$.

Since $-\Gamma^\lambda \equiv \Box x^\alpha = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\kappa} \left(\sqrt{-g} g^{\lambda\kappa} \right)$ and the Ricci Tensor,

$$-R_{\mu\nu} = \frac{1}{2} g^{\alpha\beta} \partial_\alpha \partial_\beta g_{\mu\nu} + g_{\alpha(\mu} \partial_{\nu)} \Gamma^\alpha + F_{\mu\nu}(g, \partial_g)$$

"generalized" harmonic coordinates coupled to Einstein equation gives:

$$G_{\mu\nu} = (R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R) = 8\pi T_{\mu\nu} \Longrightarrow$$

$$\frac{1}{2} g^{\alpha\beta} \partial_\alpha \partial_\beta g_{\mu\nu} + g_{\alpha(\mu} \partial_{\nu)} \Gamma^\alpha + F_{\mu\nu}(g, \partial_g) = -8\pi \left(T_{\mu\nu} - \frac{1}{2} g_{\alpha\beta} T \right) = 0 \qquad (1)$$

The gauge freedom is given by the ability to pick four arbitrary functions $\widehat{\Gamma}^\mu(g^{\alpha\beta}, x^\rho)$, set by the condition $\widehat{\Gamma}^\mu = \Gamma^\mu$.
This will provide solutions of Einstein Equations provided that the four constraints:

$$C^\mu \equiv \Gamma^\mu - \widehat{\Gamma}^\mu = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\kappa} \left(\sqrt{-g} g^{\lambda\kappa} \right) - \widehat{\Gamma}^\mu = 0$$

and their time derivatives are initially satisfied (because $\Box C^\mu = -R^\mu_\nu C^\mu$)
Any matter evolution will not affect the characteristic structure of (1).

Features of Harmonic Coordinates

- Express Einsteins equations in terms of harmonic coordinates with the addition of a source term for slicing/gauge choice freedoms.
- The constraint equations *are* the 'generalized' harmonic coordinate conditions.
- System of equations is manifestly hyperbolic (given reasonable metric conditions (one timelike coordinate, rest spacelike)).
- Simplifies the evolution equiations:
- When the gradient of this condition is substituted for terms in Einstein equations, the PP of each metric element reduces to a simple wave equation:

$$g^{\gamma\delta} g_{\alpha\beta,\gamma\delta} + \ldots = 0$$

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Thinking About Harmonic Coordinates

- There is no intuitive geometric way to think about harmonic coordinates, but we can choose a system such the we can adapt it to look like systems for which we already have some kind of understanding:
- Once the Einstein equations are recast in this way there still remains the freedom to set the rate of change of the time coordinate (lapse), and the rate of change of x^i at different along the normal to the level surfaces (shift):

Let's think about the ADM formulation as an example view of these coordinate freedoms:
n^μ is the future pointing unit normal to the slice, with $n^\mu \equiv -\alpha \nabla^\mu t$ acting as the lapse
They are related to the shift vector by $\beta^\mu \equiv t^\mu - \alpha n^\mu$ where $\beta^\mu n_\mu \equiv 0$.
The resulting spacetime metric is $ds^2 = -\alpha dt^2 + \gamma_{ij}\left(dx^i + \beta^i dt\right)\left(dx^i + \beta^j dt\right)$
The normal component and spatial projection of the source functions $\widehat{\Gamma}_\mu$ are:

$$\widehat{\Gamma}_\mu \cdot n \equiv \widehat{\Gamma}_\mu n^\mu = -K - \partial_\nu (ln\alpha) n^\nu$$

$$\perp \widehat{\Gamma}^i \equiv \widehat{\Gamma}\gamma^{\mu i} = -\Gamma^i_{jk}\gamma^{jk} + \partial_j (ln\alpha)\gamma^{ij} + \frac{1}{\alpha}\partial_{gamma}\beta^i n^\gamma$$

The time derivative of α only appears in $\widehat{\Gamma}_\mu \cdot n$ and the time derivative of β^i only appears in $\perp \widehat{\Gamma}^i$.
So evolution of $\widehat{\Gamma}_\mu \cdot n$ affects the rate of change with respect to time, and $\perp \widehat{\Gamma}^i$ affects the manner in which the spatial coordinates evolve with time.

HarmExcision

$\left(n^i D_{+i}\right)^3 \dot{f} = 0$ to all gaurd points, in layers stratified by length of the outward normal pointing vector, from out to in.

LegoExcision with excision coefficients $\frac{x^\mu}{r}$ extrapolated around a smooth virtual surface for the inner boundary.

Radiation outer boundary conditions (i.e. outgoing only).

HarmIData

- The metric and its time-derivative on the initial Cauchy hypersurface.
- The vanishing constraints and time derivatives of the constraints on the same hypersurface.
- The boundary data for the field variables and the constraints: $\partial^z \gamma^{zz} = 0, \gamma^{za} = 0, \partial^z \gamma^{ab} = 0$ where $x^a = (x, y, t)$ is well-posed and constraint preserving.

HarmEvol

- We create the evolution variables $\gamma_{\mu\nu} \equiv \sqrt{-g}g^{\mu\nu}$ and $Q_{\mu\nu} \equiv n^\rho \partial_\rho \gamma^{\alpha\beta}$, where n^ρ is future oriented and timelike. this preserves the wave equation form of the field equations.
- This simplifies the constraint equations to

$$C^\mu \equiv -\frac{1}{\sqrt{-g}} \partial_{alpha} \gamma^{\alpha\mu} - \widehat{\Gamma}^\mu$$

- The Abigel code implements the equations: $\gamma^{\alpha\beta}\partial_\alpha\partial_\beta\gamma^{\mu\nu} = S^{\mu\nu}(\gamma, \partial\gamma, \widehat{\Gamma}, \partial\widehat{\Gamma})$ rewritten in the flux conservative form: $\partial_\alpha(g^{\alpha\beta}\partial_\beta\gamma^{\mu\nu}) = \tilde{S}^{\mu\nu}$, and reduced to the first order in time system:

$$\partial_t \gamma^{\mu\nu} = -\frac{\gamma^{ti}}{\gamma^{tt}} \partial_i \gamma^{\mu\nu} + \frac{\sqrt{-g}}{\gamma^{tt}} Q^{\mu\nu} \tag{2a}$$

$$\partial_t Q^{\mu\nu} = -\partial_i \left(g^{ij} \partial_j \gamma^{\mu\nu} + g^{it} \partial_t \gamma^{\mu\nu} \right) + \tilde{S}^{\mu\nu} \tag{2b}$$

$$= -\partial_i \left(h^{ij} \partial_j \gamma^{\mu\nu} \right) - \partial_i \left(\frac{g^{it}}{g^{tt}} Q^{\mu\nu} \right) + \tilde{S}^{\mu\nu} \tag{2c}$$

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HarmEvol

- Exponential resonance mode is removes by writing evolution equations in flux conservative form and using the auxilary variable $Q_{\mu\nu} \equiv g^{t\alpha}\partial_\alpha \gamma^{\mu\nu}$
  - We find a constaint satisfying high frequency resonance mode when we write the ADM-style guage as:
  - By adding $Q_{\mu\nu} \equiv g^{t\alpha}\partial_\alpha \gamma^{\mu\nu}$ and the dnesitized metric:
- Finite differencing is 2nd or 4th order. Time integration with MoL with $T^{\mu\nu} = \partial_t \gamma_{\mu\nu}$.
  The code is evolved as a first differential order in time, second order in space system.

Gauge Conditions

- For example, let us set the source function $\widehat{\Gamma}$ to some system which evolves towards a harmonic system ($\widehat{\Gamma} = 0$):
- For these coordinates, the lapse and shift can be represented
  $\partial_t \alpha = -\alpha^2 H \cdot n + \ldots$ and
  $\partial_t \beta^i = -\alpha^2 \perp H^i + \ldots$
- $H_t(H^i)$ can be chosen to drive $\alpha(\beta^i)$ to the desired values. for example:

$$H_t = \xi \frac{\alpha - 1}{\alpha^n} \tag{3a}$$

$$\partial_t H_t = \xi \partial_t \left(\frac{\alpha - 1}{\alpha^n} \right) \tag{3b}$$

$$\nabla^\mu \nabla_\mu H_t = -\xi \frac{\alpha - 1}{\alpha^n} - \zeta \partial_t H_t \tag{3c}$$

will pull $\alpha(\beta^i)$ to one.

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Constraint Damping

- The constraint equations are the generalized harmonic coordinate conditions:

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- Dissipation: $\dot{f} \longrightarrow \dot{f} + \epsilon(\delta^{ij} D_{+i} D_{-i}) w(\delta^{ij} D_{+i} D_{-i}) f$ where w is a weight factor that vanishes at the outer boundary.

Future

- Multipatch
- SBP Boundaries
- Better initial conditions
- ...Binary Black Hole Inspiral

End.

References

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